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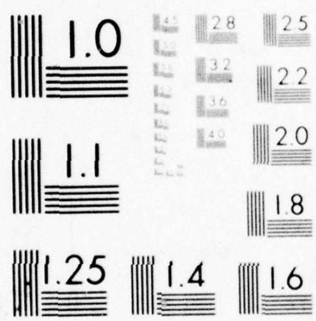

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SPARES PROVISIONING FOR REPAIRABLE ITEMS: CYCLIC QUEUES IN LIGHT TRAFFIC

by

Donald Gross
John F. Ince

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21 March 1977

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The classic machine repair problem is extended and modeled as a cyclic queue for the purpose of determining the number of spares and repair channels for a population of items subject to stochastic failure. In this system the operating units, removal of failed units, transportation to repair depot, and the repair itself are treated as four multi-server stations, each with exponential holding times. An exact model is developed from the literature on networks and cyclic queues and compared with a → next page

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1. Introduction

Reference [5] formulated a queueing model to address the problem of determining the optimum number of spares and repair channels for a population of stochastic failing units. The model assumed that a requirement for a high availability of spares was imposed and approximated the multi-stage service system with a series queue. Under the same assumptions as in Reference [5], i.e., exponential failure and service times, this paper formulates the problem as a cyclic queue for which an exact solution is tractable. This exact model can be considered as an extension of the classic machine repair problem with spares, Reference [1].

Section 2 of this paper deals with definitions and notation. The classic machine repair problem and its logical extension to many repair stages is first discussed. Then a cyclic queue is defined. Next the extended machine repair problem is framed as a cyclic queueing system, for which the literature has applicable results. The section concludes with the definition of availability.

Section 3 reviews and categorizes three key results from the literature in the field of networks and cyclic queues. Section 4 formulates both the approximate model of [5] and the exact model of this paper. Section 5 compares the accuracy and computational characteristics of the two models for gas turbine engine data from [5]. Section 6 presents the conclusions.

2. Definitions and Notation

2.1 Classic Machine Repair Problem

The classic machine repair problem with spares consists of a fixed number of identical machines of which initially M are operating and y are spares, i.e., the fixed total population is $M+y$. By identical is meant the machines have the same distributions for failure and service times, and that there are no priorities or queue disciplines other than first come, first served.

The M machines are in parallel and are independent. When one fails in service, it is instantaneously replaced by a spare, if one is available. If not, less than M machines will operate until a repaired machine becomes available. Simultaneously, the failed machine goes instantaneously into a repair facility from which, once repaired, it goes instantaneously into the spare parts pool (or directly into service, if less than M machines are operating). This process is shown in Figure 1.

The following assumptions are now made:

(a) the system failure rate is proportional to the number of operating machines,

(b) each machine has exponential failure times with mean $1/\lambda$,

(c) there are c parallel servers (repair channels) in the repair facility,

(d) each server has exponential service times with mean $1/\mu$.

Letting n equal the number of "down" machines in the repair facility, the problem becomes a Markovian birth-death process with parameters

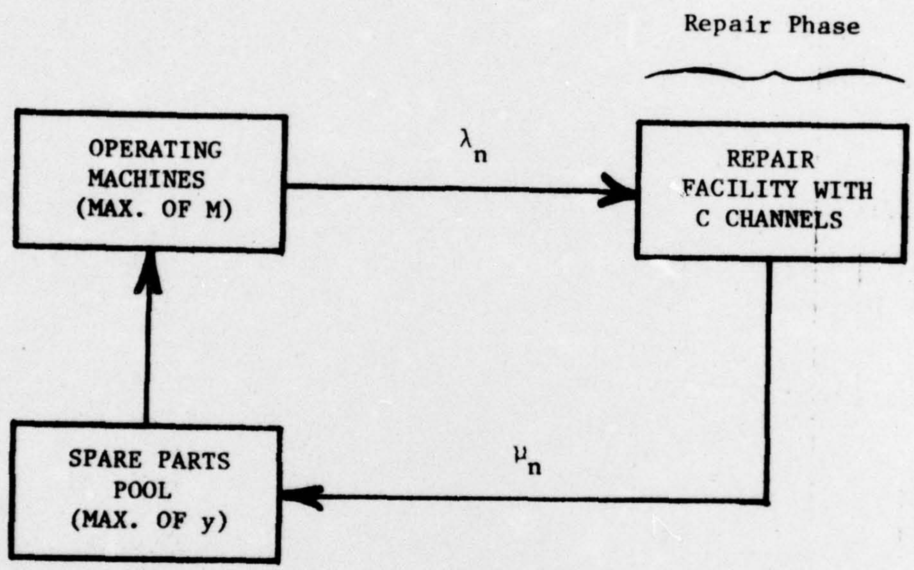

Figure 1.--Classic machine repair problem.

$$\lambda_n = \begin{cases} M\lambda & , \quad 0 \leq n < y \\ (M-n+y)\lambda & , \quad y \leq n < M+y \ , \\ 0 & , \quad n \geq M+y \end{cases}$$

and

$$_n = \begin{cases} n\mu \ , & 0 \leq n < c \\ c\mu \ , & n \geq c \end{cases} .$$

The solution for p_n , the probability that there are n machines in the repair phase, can be found on page 123 of [4].

2.2 Extension of the Machine Repair Problem to Many Stages

A logical extension to the classic machine repair problem with spares is to introduce more than one step into the repair phase. These additional steps, or stages, could represent the removal of a failed machine, the transportation to the repair facility, the repair itself, transportation from the repair facility, etc., until it is returned to the spare parts pool. Such a model is shown in Figure 2, where each additional stage consists of a number of parallel servers with exponential service times.

The straightforward application of the Markovian birth-death process is no longer directly applicable as before. Reference [5] solves this extended model by imposing a high availability constraint on the spare parts, then making a simplifying assumption that the operating stage acts as an infinite source (true Poisson) input process to a series (multi-stage repair) queue. The model of this paper solves the extended model by treating it as a cyclic queue, without any requirement for high availability.

2.3 Cyclic Queueing System

Figure 3 represents a cyclic queueing system. There is a total of N identical customers in the K separate stages of the system. Each stage consists of c_i parallel servers, each with exponentially distributed service times with mean μ_i , $i=1,2,\ldots,K$. The system is closed, i.e., no

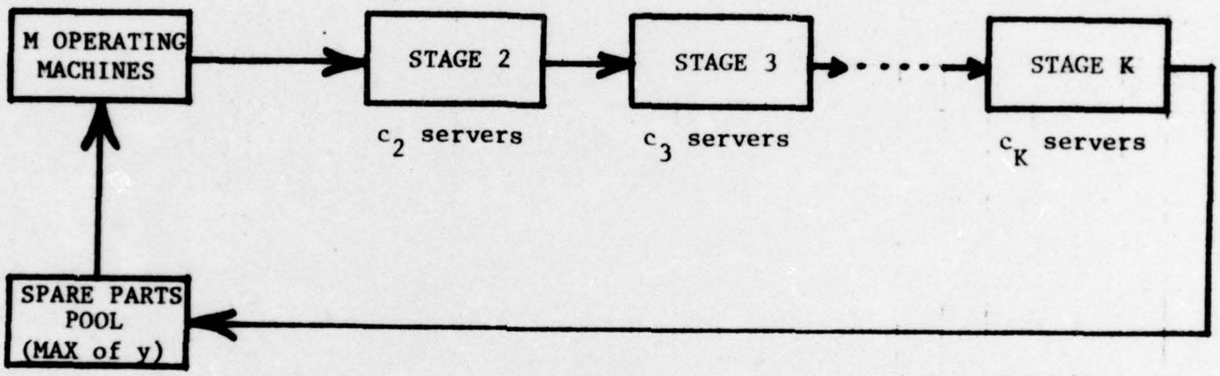

Figure 2.--Extension of classic machine repair problem.

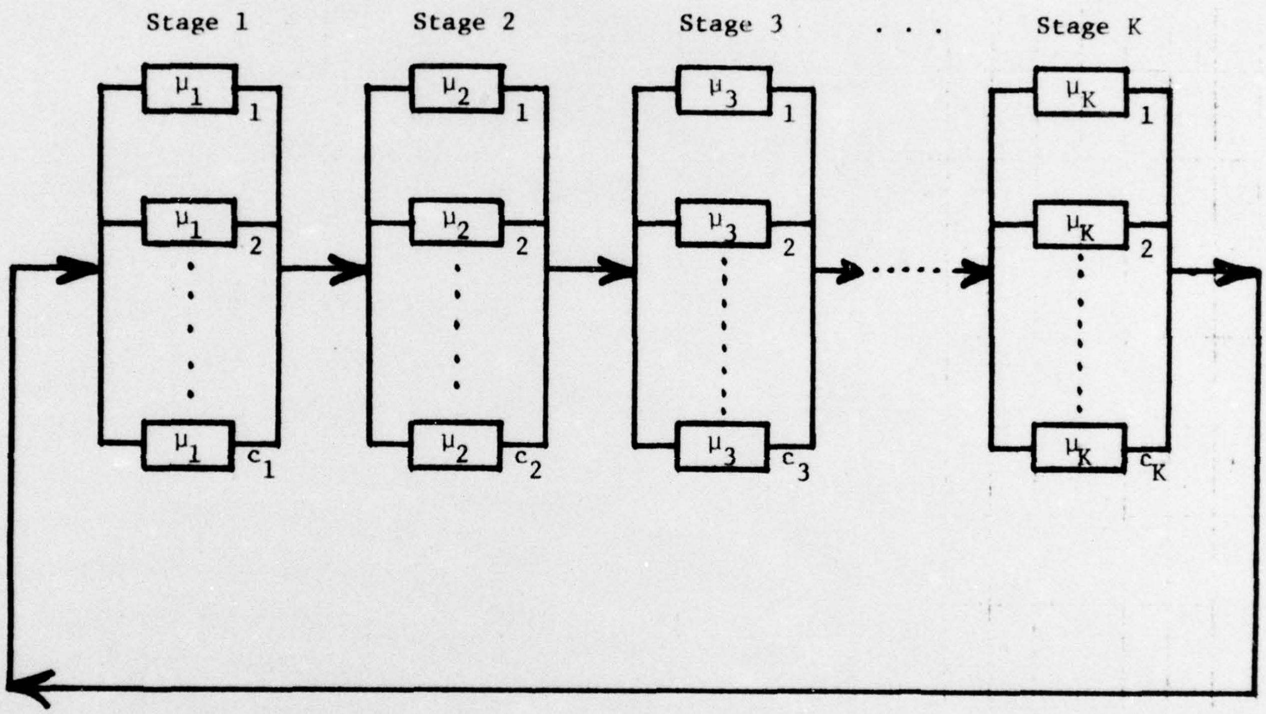

Figure 3.--Cyclic queueing system.

customers enter or leave the system. Upon being served, a customer goes directly to the next stage. If a server is free, the customer goes directly into service. If not, the customer forms or joins a queue, which can never be blocked (i.e., infinite waiting room between stages). Upon completing the Kth stage, the customer goes directly to the first stage and starts the cycle over.

Letting n_i , c_i , and μ_i represent the number of customers, parallel servers, and exponential service mean rate, respectively, at the ith stage, $i=1,2,\ldots,K$, we have the following relationships:

$$\text{service rate at ith stage} = \begin{cases} n_i\mu_i , & n_i \leq c_i \\ c_i\mu_i , & n_i > c_i \end{cases}$$

and

$$\text{total number of customers in the } K \text{ stages of the system} = N = \sum_{i=1}^{K} n_i .$$

2.4 Machine Repair Problem as a Cyclic Queue

The extended machine repair problem with spares will now be framed as a cyclic queue. Of the K stages of the cyclic queue, the first stage represents the operating machines. The c_1 parallel servers at the first stage can be considered to be the M operating machines of the classic problem $(c_1 = M)$. Machines that fail in Stage 1 go directly to Stage 2, a removal phase. There are c_2 parallel servers there which represent the number of "machine removers" present. After removal, machines go to Stage 3, say a transportation phase, then to Stage 4, etc. At the ith stage $(i=1,2,\ldots,K)$, there are c_i parallel servers, each with an exponential service rate μ_i . Note that c_i can be set to the total population size, N , in the system (i.e., effectively set to infinity) to represent an ample server stage, with no possibility of a queue forming in front of that stage. This might be appropriate for removal and transportation phases.

After leaving the Kth stage, the machine returns to Stage 1 ready for service. If less than c_1 (i.e., M) machines are operating, the newly repaired machine goes right into service. If all c_1 servers are busy (i.e., all M are "up") the newly repaired machine either starts a queue or joins an existing one in front of Stage 1. This queue represents the spare machines on hand (i.e., the inventory).

Table 1 compares the terminology between the machine repair problem and the cyclic queueing system.

TABLE 1

COMPARATIVE TERMINOLOGY

NOTATION	MACHINE REPAIR PROBLEM	CYCLIC QUEUE
Number of Operating Machines	M	c_1
Machine Failure Rate	λ	μ_1
Total Number of Machines	M+y	N
Number of Spares	y	$N - c_1$
Size of Inventory (Spare Parts Pool)	M + y - Number in Repair Phase	Queue Size at Stage 1

2.5 Definition of Availability

Suppose there are three spare machines in the inventory, i.e., a queue of size three in front of Stage 1. If an operating machine fails, a spare is instantaneously pulled from the inventory and put into service. Therefore, at the time of a failure, a spare is available. Now assume that the queue has shrunk to one spare. Again, if an operating machine fails, a spare is available. At this point, the queue size is zero and no spares are available, but the operating system is unconcerned; the operating system looks to the inventory only when an operating machine fails, and at no other time. By the time another machine fails, a newly repaired machine might have arrived from the Kth phase, resulting in a positive queue again. In other words, even though temporarily no spares were available, the operating system does not know or care about it unless a failure occurs. As long as M machines are up without interruption, the availability of spares is not a factor in the operation. Therefore, the definition of availability involves conditioning on the occurrence of a failure. More precisely,

AVAILABILITY = probability that the spares inventory is not empty given that a failure is about to take place

= probability that the queue size at Stage 1 is greater than 0 given that a failure is about to take place.

An algebraic expression for these "failure point probabilities" in terms of the "general time probabilities" can be derived from Bayes' theorem. First, define[1]

$P(n_1)$ = general time probability that there are n_1 customers at Stage 1

and

$Q(n_1)$ = conditional (or failure point) probability that there are n_1 customers at Stage 1 given that a failure is about to occur.

By Bayes' theorem:

$$Q(n_1) = \frac{\Pr\{\text{failure about to occur at Stage } 1 \mid n_1 \text{ customers at Stage } 1\} \cdot P(n_1)}{\sum_{n_1=0}^{N} \Pr\{\text{failure about to occur} \mid n_1\} \cdot P(n_1)}, \tag{1}$$

---

[1]All probabilities are assumed to be steady-state.

where N is the maximum number of customers in the system. We will return to Equation (1) in the development of the specific models.

With $Q(n_1)$ so defined, we can now define availability as:

$$\text{AVAILABILITY} = \sum_{n_1=c_1+1}^{N} Q(n_1) \ .$$

It remains now to find the $P(n_1)$ using the theory of cyclic queues.

3. Literature Review on Cyclic Queues and Networks

In the course of research for this paper more than 40 references on the subject of cyclic queues and networks were reviewed and categorized. Networks were included because cyclic queues can be considered as a subset of networks. Only the three most relevant references to this paper's application are discussed here. The others, dealing with variations such as travel time between stages (see [9]), are not included.

Table 2 shows the key features of the models presented in the three pertinent references. All of the models are multi-stage, all stages have only exponential service, there is no travel time between stages, and all customers are identical (i.e., single class of customer). The no travel time restriction is not crucial since travel time between stages can be handled by simply introducing an ample server transportation stage between any two stages in the cyclic queue.

3.1 J. R. Jackson, Reference [6]

In his 1957 paper, Jackson proved a theorem for networks in a steady-state condition. The theorem yields the steady-state joint probability for the number of customers at each stage. The network has multiple stages. Each stage has parallel channels, with all servers at a given stage having the same exponential service time distribution. Each stage can also have external Poisson input to it and can output from the system. Customers could go from one stage to any other stage in the network (feedback/feedforward)

TABLE 2

LITERATURE ON CYCLIC QUEUES AND NETWORKS

FEATURE / REFERENCE	J. R. JACKSON 1957 [6]	E. KOENIGSBERG 1958 [8]	R. SWERSEY 1967 [10]
System	Network	Cyclic Queue	Cyclic Queue
Number of Stages	Multiple	Multiple	Multiple
Service Distribution	Exponential	Exponential	Exponential
Service Mean	$\left.\begin{matrix} n\mu \\ c\mu \end{matrix}\right\}$ Parallel Channel	μ , Single Server	$\mu(n)$, State Dependent
Exogenous I/O	Yes, Exponential	No	No
Feedback/Feedforward	Yes	No	No
Travel Time	No	No	No
Different Classes	No	No	No

according to some known probability distribution. In 1963 [7], Jackson extended his theorem to allow state-dependent service at any stage.

Before describing the Jackson results, we will first consider a single M/M/c queue, with input Poisson stream parameter λ and exponential service parameter μ . From the well-known results for p_n , the steady-state probability that there are n customers in the system (service plus queue), given the parameters λ , μ , and c , are:

$$p_n(\lambda,\mu,c) = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} p_0 & ; \quad n \leq c \\ \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!\, c^{n-c}} p_0 & ; \quad n > c \, , \end{cases} \tag{3}$$

$$(\text{for } \frac{\lambda}{c\mu} < 1) \, .$$

Jackson considers a network such as the one in Figure 4, where a customer's path through the network is influenced by p_{ji} , the probability that a customer leaving the jth stage goes to stage i ($j=1,2,\ldots,K$, $i=0,1,\ldots,K$) ; $i=0$ represents leaving the system and $\sum_{i=0}^{K} p_{ji} = 1$ for all j . He defines α_i as the Poisson parameter of the external input to stage i, and Γ_i to be the total mean input rate to stage i. Therefore,

$$\Gamma_i = \alpha_i + \sum_{j=1}^{K} \Gamma_j p_{ji} \, . \tag{4}$$

Essentially, Jackson's theorem states that the steady-state joint probability of n_1 customers at Stage 1, n_2 at Stage 2, ..., n_K at Stage K, can be determined by first assuming that each stage is an independent M/M/c queue with input parameter Γ_i and service parameter rate μ_i , then using Equation (3) at each stage, and then multiplying the results to obtain the joint probability distribution. This can be written as (for $\Gamma_i < \mu_i c_i$, for all $i=1,\ldots,K$)

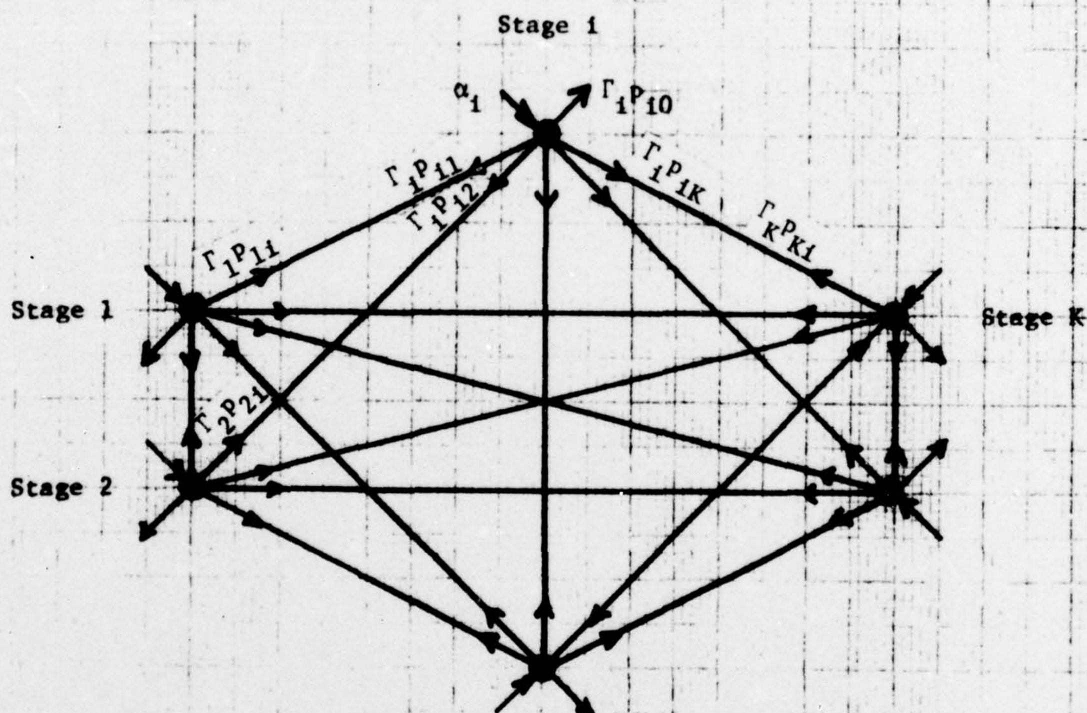

Figure 4.--Jacksonian network.

$$p_{n_1,n_2,\ldots,n_k}(\Gamma_i,\mu_i,c_i,\ i=1,\ldots,K)$$

$$= p_{n_1}(\Gamma_1,\mu_1,c_1)p_{n_2}(\Gamma_2,\mu_2,c_2)\ldots p_{n_K}(\Gamma_K,\mu_K,c_K)\ , \qquad (5)$$

where $p_{n_i}(\Gamma_i,\mu_i,c_i)$ is determined from Equation (3) with the parameters substituted accordingly.

Note that Jackson's theorem does not assert independence of the stages, just that if independence is assumed, the resulting joint probability is correct. In his proof, he first sets up the steady-state difference equations, then postulates the solution, and finally shows that the solution satisfies the equations.

3.2 E. Koenigsberg, Reference [8]

In his 1958 paper, Koenigsberg looked at a cyclic queue with multiple stages, but with only a single server at each stage. The service times were exponential. Since it was a cyclic queue and not a network, there was no exogenous input or output and no feedback or feedforward, except directly to the next stage.

The method of Koenigsberg was to set up the differential difference equations for all possible states. The resulting set of $\binom{N+K-1}{K-1}$ equations in the same number of unknowns (which is the number of ways to put N indistinguishable items into K boxes, any number to a box), was intractable for a general solution. Koenigsberg then postulated the solution, which satisfied the set of simultaneous state equations when substituted back in. Were it not for the limitation to a single server at each state, we could have used Koenigsberg's results directly for the development of an exact model for the extended machine repair problem.

3.3 R. Swersey, Reference [10]

Swersey noted in his 1967 paper that cyclic queues are subsets of networks and he applied Jackson's network results of [6] and [7] to a

Koenigsberg type of cyclic queue. This amounts to setting α_i to zero for all i and p_{ji} to one for $j=i-1$ and to zero otherwise in Equation (4). Therefore, $\Gamma_i = \Gamma_{i-1}$ for all i, $i=1,2,\ldots,K$ (with the understanding that $\Gamma_0 = \Gamma_K$). Thus, Γ is a constant (to be determined) throughout the system. This means that in steady state, the input rate to any stage is a constant, and if the cyclic queue is cut at any point between any two stages, the same constant flow rate of customers, Γ, would be observed.

Under the restrictions that the sum of customers is constant (N) and that the joint probability must integrate to unity, the Jackson theorem, as applied by Swersey, can be used to solve for steady-state probabilities for cyclic queues.

4. Formulation of the Exact and Approximate Models

4.1 The Exact Model

By the Jackson theorem, Equation (5) gives the steady-state joint probability where Equation (3) yields the individual factors in the expression.

Applying the Swersey analysis (i.e., Γ a constant) yields

$$p_{n_1,n_2,\ldots,n_K} = \left(\frac{\Gamma}{\mu_1}\right)^{n_1} \frac{1}{b_1} \cdot \left(\frac{\Gamma}{\mu_2}\right)^{n_2} \frac{1}{b_2} \cdot \ldots \cdot \left(\frac{\Gamma}{\mu_K}\right)^{n_K} \frac{1}{b_K} \cdot p_{0_1} \cdot p_{0_2} \cdot \ldots \cdot p_{0_K}$$

$$= \left\{\frac{\Gamma^N}{\mu_1^N} \prod_{i=1}^{K} p_{0_i}\right\}\left\{\prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i}\right\},$$

where $p_{n_1,\ldots,n_K}$ is the steady-state joint probability that there are n_i customers at Stage i, for i going from 1 to K. For $i=1,2,\ldots,K$:

$$b_i = \begin{cases} n_i! & ; \quad n_i \le c_i \\ c_i!\, c_i^{n_i-c_i} & ; \quad n_i > c_i \end{cases}$$

n_i = number of customers at Stage i ($\sum_{i=1}^{K} n_i = N$)

c_i = number of parallel servers at Stage i

μ_i = mean service rate of each server at Stage i

p_{0_i} = steady-state probability that there are zero customers at the ith stage when the ith stage is treated as an independent $M/M/c_i$ queue with Poisson input Γ and service rate μ_i .

Since the first factor does not depend on n_i , it can be found by the summation to unity criterion. Denoting it by A_1 , we can write

$$p_{n_1,n_2,\ldots,n_K} = A_1 \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} , \tag{6}$$

where A_1 , then, is given by

$$A_1^{-1} = \sum_{S_1} \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} , \tag{7}$$

and

$$S_1 = \{n_i,\ i=1,2,\ldots,K : \sum_{i=1}^{K} n_i = N\} . \tag{8}$$

In the above "constant of integration," A_1 , the summation is taken over all possible partitions of the N customers into the K stages. Note that the joint probability in Equation (6) no longer involves the unknown steady-state system flow Γ , since it has been incorporated into the constant of integration.

To determine the marginal probability for the number of customers at Stage 1, n_1 , we must sum the joint probability over all partitions of $N-n_1$ into $K-1$ stages, i.e., over the set S_2 , where

$$S_2 = \{n_i,\ i=2,3,\ldots,K : \sum_{i=2}^{K} n_i = N-n_1\} . \tag{9}$$

Letting $P_E(n_1)$ represent the steady-state "exact" (since we will be comparing this exact model to an approximate model mentioned later) marginal probability that there are n_1 customers at Stage 1,

$$P_E(n_1) = A_1 \sum_{S_2} \left\{ \prod_{i=2}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} . \tag{10}$$

To determine the availability of spares, we must first determine the failure point probability, Equation (1). (Note that $P_E(n_1)$, Equation (10), is a general time probability.) Since we have an exponential service discipline,

Pr{failure about to occur at Stage 1 | n_1 customers at Stage 1}

$$= \begin{cases} n_1\mu_1\Delta t + o(\Delta t) \ ; & n_1 \le c_1 \\ c_1\mu_1\Delta t + o(\Delta t) \ ; & n_1 > c \end{cases} .$$

Equation (1) can now be written (with subscript E added to denote this exact model) after dividing numerator and denominator by Δt and taking $\lim_{\Delta t \to 0}$, as

$$Q_E(n_1) = \begin{cases} A_3 n_1 \mu_1 \ P_E(n_1) \ ; & n_1 \le c_1 \\ A_3 c_1 \mu_1 \ P_E(n_1) \ ; & n_1 > c_1 \end{cases} , \tag{11}$$

where

$$A_3^{-1} = \sum_{n_1=0}^{c_1} n_1\mu_1 \ P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1\mu_1 \ P_E(n_1) . \tag{12}$$

Equation (2), the expression for the availability, now becomes (again with the subscript E added)

$$\text{AVAIL}_E = \sum_{n_1=c_1+1}^{N} Q_E(n_1) = A_3 c_1 \mu_1 \sum_{n_1=c_1+1}^{N} P_E(n_1) . \tag{13}$$

Equation (13) is the availability of the system. It requires Equations (12), (10), and (7) to compute, but the availability can be expressed totally in terms of the system parameters μ_i, c_i $(i=1,\ldots,K)$, N , and K .

Note that the constant A_3 , in Equation (12), requires the evaluation of $P_E(n_1)$ for all n_1 from 0 to N, the maximum in the system. This is a key point in discussing the computational efficiency of the exact and approximate models.

4.2 The Approximate Model

Consider an M/M/c queue with input parameter λ and service rate μ . The steady-state probability that there are n in the system is given by Equation (3). Burke, in his 1956 paper [2], proved that the output from that queue was Poisson with parameter λ . Thus, the output is independent of the service rate mean, μ , as long as the service time is exponentially distributed. Burke also reasoned in his 1972 paper [3], that since λ in results in λ out, then a series of M/M/c queues could be formed with λ in at one end and λ out at the other, as indicated in Figure 5. Each stage is independent of the other stages and the steady-state joint probability could be found by multiplying the probability from Equation (3) for each stage.

The crux of the approximate model is the assumption that the first stage, the operating machine stage, is almost always operating at full capacity, i.e., all c_1 machines are up. (This is the implication of a high availability constraint.) Then the output from the first stage is a pure Poisson process with parameter $c_1\mu_1$ and the first stage acts like an infinite source input to the rest of the system.

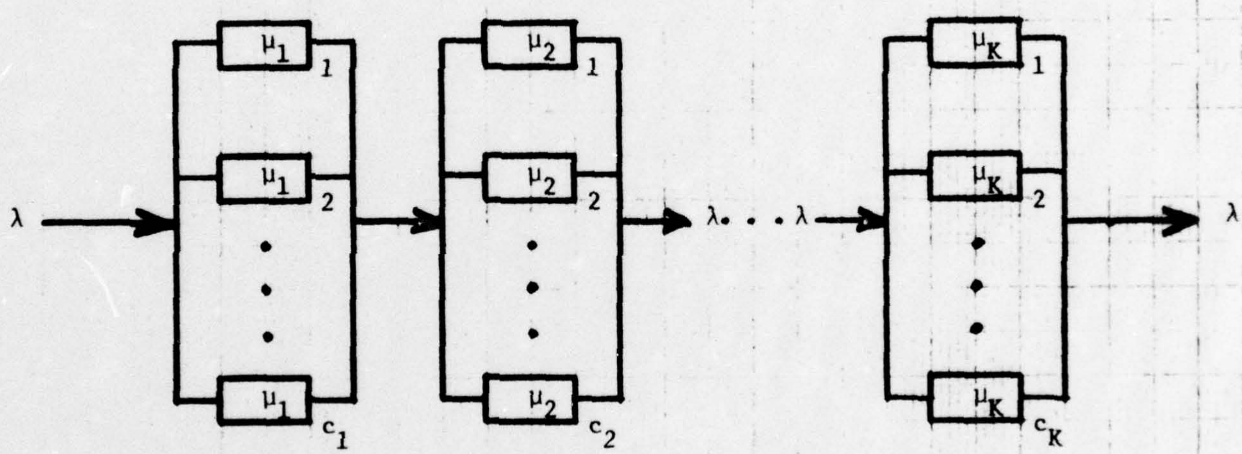

Figure 5.--Burke series queue.

Stages 2 through K can now be treated as the Burke series of Figure 5, and the joint probability that there are n_2 in Stage 2, n_3 in Stage 3, ..., n_K in Stage K is

$$p_{n_2,n_3,\ldots,n_K} = \left(\frac{c_1\mu_1}{\mu_2}\right)^{n_2} \frac{1}{b_2} p_{0_2} \cdots \left(\frac{c_1\mu_1}{\mu_K}\right)^{n_K} \frac{1}{b_K} p_{0_K} , \tag{14}$$

where for $i=2,\ldots,K$

$$b_i = \begin{cases} n_i! & ; \quad n_i \le c_i \\ c_i!\, c_i^{n_i-c_i} & ; \quad n_i > c_i \end{cases} ,$$

$c_1\mu_1 < c_i\mu_i$ for $i=2,\ldots,K$ and p_{0_i} = probability there are zero customers at the ith stage.

After summing over all possible n_i to get the "constant of integration,"

$$p_{n_2,n_3,\ldots,n_K} = A_2 \prod_{i=2}^{K} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} , \tag{15}$$

where

$$A_2^{-1} = \prod_{i=2}^{K} \sum_{n_i=0}^{\infty} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} = \prod_{i=2}^{K} \left\{ \sum_{n_i=0}^{c_i-1} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{n_i!} + \frac{1}{c_i!}\left(\frac{c_1\mu_1}{\mu_i}\right)^{c_i} \left(\frac{c_i\mu_i}{c_i\mu_i - c_1\mu_1}\right) \right\} . \tag{16}$$

At this point there is no restriction on $\sum_{i=2}^{K} n_i$ because with an infinite source at Stage 1, any system size is possible. However, in the real environment, there is a constraint, namely $\sum_{i=2}^{K} n_i = N-n_1$, where N is the total population in the system and n_1 is the number at Stage 1, the "infinite source." Therefore, the probability that there are n_1 at Stage 1 is set equal to the probability that $N-n_1$ are in the rest of the system, i.e., the sum of the probabilities of all the possible ways $N-n_1$ customers can be partitioned into $K-1$ stages.

Letting $P_A(n_1)$ be the approximate model's steady-state probability that there are n_1 at Stage 1, we have

$$P_A(n_1) = A_2 \sum_{S_2} \prod_{i=2}^{K} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} , \tag{17}$$

where

$$S_2 = \{n_i,\ i-2,3,\ldots,K : \sum_{i=2}^{K} n_i = N-n_1\} ,$$

A_2 is given by Equation (16), and Equation (15) has already been substituted for the joint probability function.

It is well known that for M/M/c queues,

$$Q_A(n_1) = P_A(n_1) . \tag{18}$$

By the definition of availability, Equation (2), we can now write (with the subscript A added to denote the approximate model)

$$\begin{aligned} \text{AVAIL}_A &= \sum_{n_1=c_1+1}^{N} Q_A(n_1) , \\ &= \sum_{n_1=c_1+1}^{N} P_A(n_1) . \end{aligned} \tag{19}$$

Equation (19) is the availability of the system. It requires Equations (17) and (16) to compute, but the availability can be expressed totally in terms of the system parameters μ_i , c_i $(i=1,\ldots,K)$, N , and K .

Note that Equation (19) requires only the evaluation of $P_A(n_1)$ for $n_1 > c_1$. This is a key point in discussing the computational efficiency of the exact and approximate models.

4.3 Summary of Algebra and Some Inequalities

Exact model: The exact, marginal, general time probability that there are n_1 customers at the first stage is

$$P_E(n_1) = A_1 \sum_{S_2} \left\{ \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\}, \qquad (10)$$

where

$$A_1^{-1} = \sum_{S_1} \left\{ \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\}, \qquad (7)$$

$$S_2 = \{n_i,\ i=2,\ldots,K : \sum_{i=2}^{K} n_i = N-n_1\} ,$$

$$S_1 = \{n_i,\ i=1,2,\ldots,K : \sum_{i=1}^{K} n_i = N\} ,$$

$$b_i = \begin{cases} n_i! & ; \quad n_i \le c_i \\ c_i!\ c_i^{n_i-c_i} & ; \quad n_i > c_i \end{cases} .$$

The exact, failure point probability that there are n_1 customers at the first stage given that a failure is about to occur is

$$Q_E(n_1) = \begin{cases} A_3\ n_1\mu_1\ P_E(n_1) \ ; & n_1 \le c_1 \\ A_3\ c_1\mu_1\ P_E(n_1) \ ; & n_1 > c_1 \end{cases} , \qquad (11)$$

where

$$A_3^{-1} = \sum_{n_1=0}^{c_1} n_1\mu_1 \; P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1\mu_1 \; P_E(n_1) \; . \qquad (12)$$

The probability that there are spares available given that a failure is about to occur is

$$\mathrm{AVAIL}_E = \sum_{n_1=c_1+1}^{N} Q(n_1) = A_3 \; c_1\mu_1 \sum_{n_1=c_1+1}^{N} P_E(n_1) \; . \qquad (13)$$

Approximate model: The approximate, marginal, general time probability that there are n_1 customers at the first stage is

$$P_A(n_1) = A_2 \sum_{S_2} \left\{ \prod_{i=2}^{K} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\}, \qquad (17)$$

where

$$A_2^{-1} = \prod_{i=2}^{K} \left\{ \sum_{n_i=0}^{\infty} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\}, \qquad (16)$$

and S_2 , b_i as above in the exact model.

The approximate failure point probability that there are n_1 customers at the first stage given that a failure is about to occur is

$$Q_A(n_1) = P_A(n_1) \; . \qquad (18)$$

The probability that there are spares available given that a failure is about to occur is

$$\mathrm{AVAIL}_A = \sum_{n_1=c_1+1}^{N} P_A(n_1) \; . \qquad (19)$$

Inequalities: With the above algebra summarized, it can be shown (see Appendix) that for $n_1 > c_1$,

$$Q_E(n_1) \geq P_E(n_1) \geq Q_A(n_1) .$$

Therefore,

$$\sum_{n_1=c_1+1}^{N} Q_E(n_1) \geq \sum_{n_1=c_1+1}^{N} P_E(n_1) \geq \sum_{n_1=c_1+1}^{N} Q_A(n_1) ,$$

or

$$AVAIL_E \geq AVAIL_{GE} \geq AVAIL_A , \tag{20}$$

where

$$AVAIL_{GE} = \sum_{n_1=c_1+1}^{N} P_E(n_1) , \tag{21}$$

and where the subscript GE represents an availability determined by using the general time probability (versus a failure point probability) from the exact model.

Inequality (20) is a key result. It shows that for a given set of system parameters $(N, c_i, \mu_i, i=1,\ldots,K)$, the true (exact) availability of the system is greater than that predicted by the approximate model or by using the general time probability (versus the failure point) of the exact model.

In other words, the approximate and GE availabilities are conservative, i.e., if one uses the approximate or GE approach to determine system parameters to yield a certain availability, then the true system availability will actually be greater than the original target.

4.4 Relationship Between the Two Models

The exact model development involves the concept of a fixed flow rate Γ throughout the cycle. Consider the flow coming out of the first stage. If there is only one customer in service, the output rate from the stage is equal to the service rate, μ_1 , for one channel. If two are in service, the output rate is $2\mu_1$. The output rate increases linearly with the number of customers in service, up to the maximum service rate of

$c_1\mu_1$. These different output rates, weighted by the probability of each rate, are what make up Γ .

In the exact model,

$$\Gamma = \sum_{n_1=0}^{c_1} n_1\mu_1 \; P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1\mu_1 \; P_E(n_1) \; ,$$

or, after some algebra,

$$\Gamma = c_1\mu_1 - \sum_{n_1=0}^{c_1} (c_1-n_1)P_E(n_1) \; , \tag{22}$$

The second term is negligible under a high availability constraint (i.e., there is probably a queue at Stage 1, which implies n_1 is probably greater than c_1 , which implies $P_E(n_1)$ small for $n_1 \leq c_1$). Therefore, under a high availability constraint,

$$\Gamma \approx c_1\mu_1 \; , \tag{23}$$

which is a pure Poisson output with parameter $c_1\mu_1$. This is what the approximate model uses as output from its first stage ("infinite source") and input to its second stage.

In summary, the exact model uses Γ from Equation (22) and the approximate model uses Γ from Equation (23).

5. Computations

5.1 Data

The following data were obtained from [5]. The application was to a fleet of gas turbine engine ships, where the decision variables were the number of spare engine parts to supply and the number of repair channels to provide in order to minimize costs and to satisfy a high (.90) availability constraint. Each engine had two components, a gas generator and a power turbine, requiring separate repair facilities; however, only one component will be analyzed here. There are four stages: operating, removal, transportation, and repair. Two cases are examined, one where it is desired to have 10 operating engines, and the other where it is desired to have 28 operating engines. The removal and transportation stages have ample servers, i.e., no queues ever form in front of Stages 2 and 3. The parameters for the gas generator component (the more critical of the two) are given below:

μ_1 = failure rate = .00147186 failures/day

μ_2 = removal rate = .5 removals/day

μ_3 = transportation rate = .1 transports/day

μ_4 = repair rate = .01887 repairs/day

or, in terms of the inverses,

$1/\mu_1$ = 679.4 days between failures

$1/\mu_2$ = 2 days to remove a machine

$1/\mu_3$ = 10 days to transport a machine

$1/\mu_4$ = 53 days to repair a machine

c_1 = 10 ; 28 = number of operating machines

c_2 = c_3 = "∞" = ample servers for removal and transportation.

Decision variables:

c_4 = number of repair channels at the repair facility

N = total number of customers in the system = number of spares + c_1 .

Reference [5] defined an objective function using cost data to arrive at optimal values for the decision variables (number of spares, number of repair channels). In this paper no objective function will be formed. Only the effect of varying the decision variables on the availability for the exact and approximate models will be analyzed for the purpose of determining the accuracy of the approximate model.

5.2 Discussion of Results

Figure 6 shows the results for the case with 10 operating engines ($c_1 = 10$) . The abscissa is the number of repair channels, c_4 , at the fourth stage (Stages 2 and 3 have ample, or infinite, servers), and the ordinate is the availability of spares.

The curves plotted represent the availability as a function of c_4 for a fixed number of spares. The availability is computed three ways: by the exact model, $AVAIL_E$, Equation (13); by the approximate model, $AVAIL_A$, Equation (19); and by the general time probability of the exact model, $AVAIL_{GE}$, Equation (21). Note how the computed availabilities are consistent with Inequality (20), which shows the GE and approximate models to be conservative.

Just to the right of the model designation ($AVAIL_E$, etc.) is the computer time and percentage error. The time is the average number of "system seconds" (a combination of compilation, execution, input/output, etc.) of computer time used in the calculation of any given point on that curve. The percentage error is relative to the exact model and is computed after the apparent asymptote has been reached. Percentage errors are also shown at two other points on each curve, before the asymptote is reached.

For example, for the one spare case, the approximate model took about 3.1 system seconds to compute each point on the curve (which is composed of about five points), and the error was about 4% for points with more than three servers. This error increased to 6% for two servers, and to 36% for one server. Note that the number of servers is with respect to the fourth stage of this cyclic queueing system.

The results indicate that the availabilities quickly approach horizontal asymptotes as the number of servers increases. In addition, the asymptotes for E, GE and A more quickly approach each other as the number of spares increases, i.e., the percentage error from not using the exact model decreases.

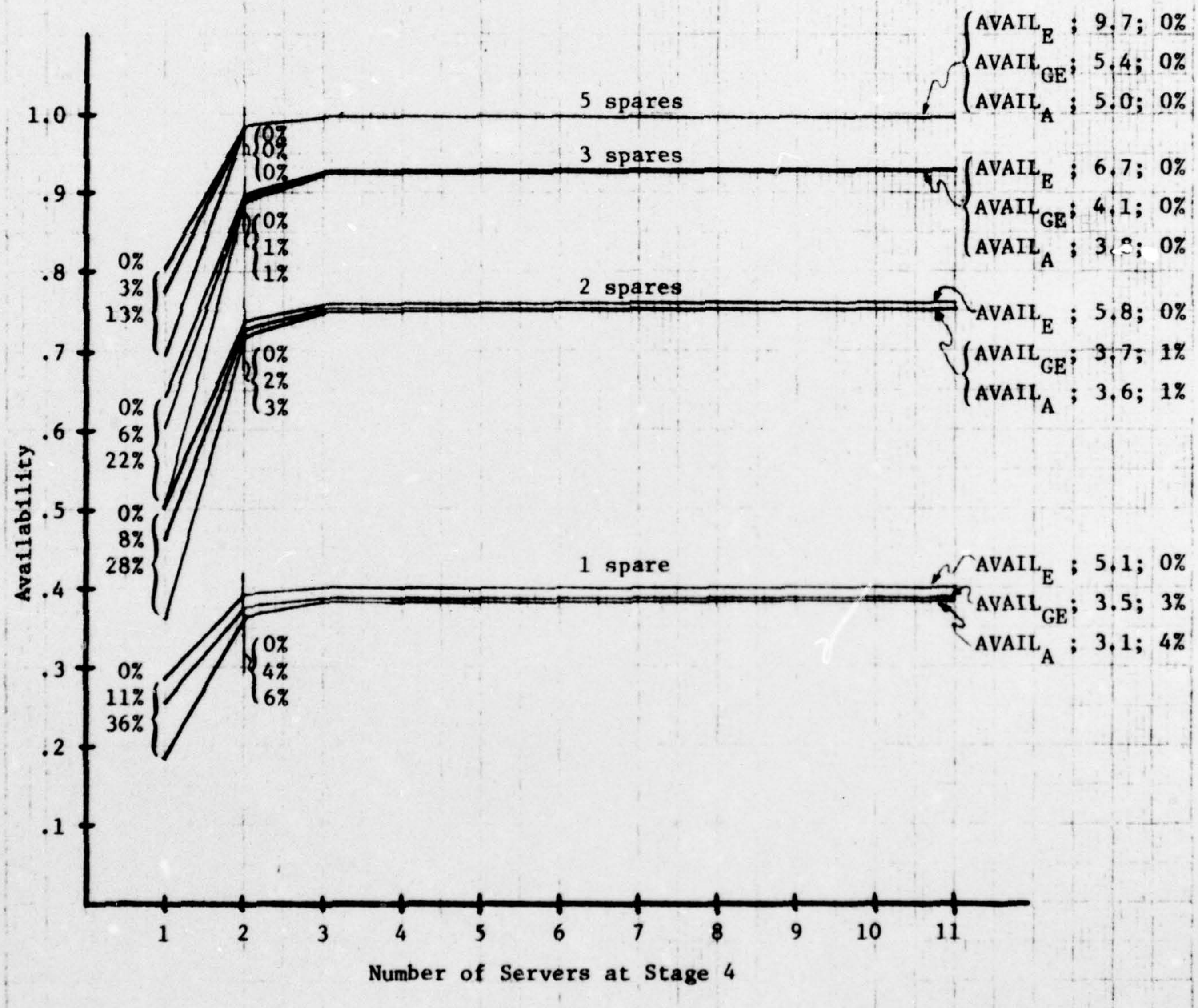

Figure 6.--10 operating machines ($M = c_1 = 10$).

A key observation is that if the availability were constrained to be "high," say .9, then the objective function for a particular application would only be evaluated over a region where the number of servers is greater than two and the number of spares is greater than three. In this region, the approximate and exact models, as well as the exact general time probability approach, yield almost the same results.

In Figure 7 we have similar trends for the case where $c_1 = 28$ (i.e., 28 machines are to be operating). Here an availability constraint of .9 would force the objective function to a range in which there are at least four servers and six spares. Again, the approximate model and the exact general time approach yield very good results with respect to the exact model. However, note how the computer times vary. The approximate model takes only about one-sixth of the time for almost the same accuracy.

This time difference can be explained by looking at the algebraic summary, Section 4.3. It takes more time to compute the constant of integration, A_1, Equation (7), for the exact model (versus A_2, Equation (16), of the approximate model), but this is but a small part of the difference. Equation (13) for the exact model requires the computation of a constant A_3, Equation (12), which involves $P_E(n_1)$ for all n_1 from 0 to N, the maximum in the system. On the other hand, the approximate model only involves $P_A(n_1)$ for n_1 from c_1+1 to N in Equation (19). Considering that for each value of n_1 both the exact and the approximate models must go through the partitioning of $N-n_1$ customers into K-1 subsets (i.e., the set S_2 in Equations (10) and (17)), the approximate model becomes increasingly easier than the exact model to evaluate as N increases (where $N = c_1$ + number of spares). Thus, for $c_1=10$ (and N ranging from 11 to 15) the time differences were relatively great, but in magnitude not startling; whereas for $c_1=28$ (and N ranging from 30 to 34), the computation time differences were significant, both relatively and absolutely.

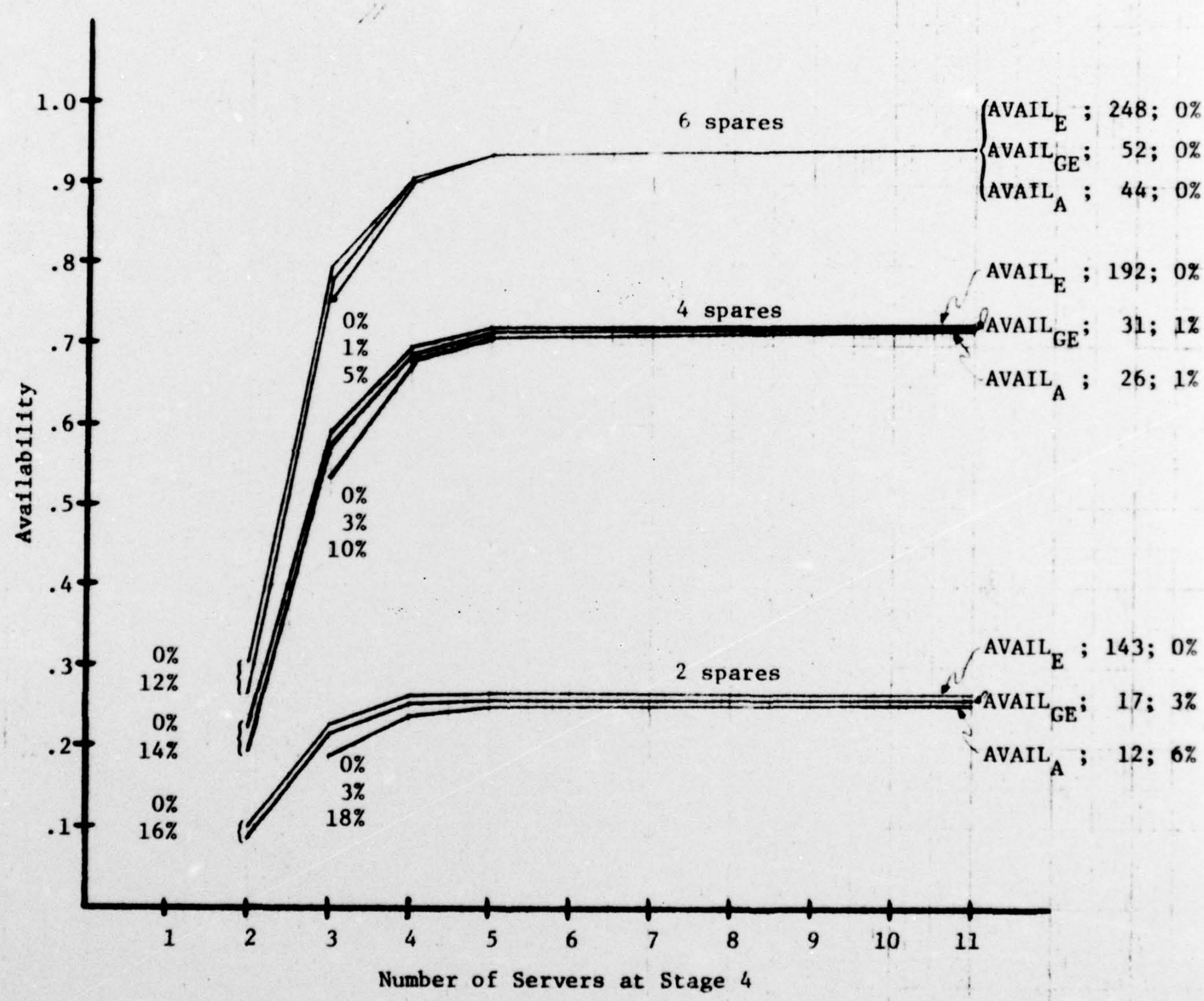

Figure 7.--28 operating machines ($M = c_1 = 28$).

When comparing the approximate model with the general time probability of the exact model in Equation (21) (the GE model), the approximate model is faster than the GE model due to the time involved in computing the constants A_2 and A_1, Equations (16) and (7), respectively. However, this general time probability from the exact model approach still yields substantial time savings over the exact model because the constant A_3, Equation (12), is not involved in the GE model.

Some observations on the asymptotic behavior of availability are worth noting. Figures 6 and 7 clearly indicate that availability can be increased only to a certain value as the number of servers at Stage 4 is increased (for a constant number of spares). Obviously, when the number of servers exceeds the total population in the system, the availability can no longer be affected. But the asymptote is reached long before the number of servers becomes "ample" at Stage 4.

There appears to be a similar asymptote for availability given a fixed number of servers and an increasing number of spares (i.e., a vertical cut in Figures 6 and 7 to derive a cross plot). If contours of constant availability in server-spare space are plotted, a picture such as Figure 8 results.

One final note on Figure 7: for less than three servers at Stage 4, the approximate model cannot be used because the condition $c_1\mu_1 < c_i\mu_i$ for $i=2,\ldots,K$ is not satisfied. Thus, the approximate model has a limit on its applicability. However, the approximate model was formulated with a high availability constraint at Stage 1 in mind. For $c_1\mu_1 > c_4\mu_4$ there is great congestion at Stage 4 rather than at Stage 1, and the high availability constraint at Stage 1 is not satisfied.

6. Conclusions

This paper has formulated and compares an exact and approximate model to handle an extended machine repair with spares problem. Under a

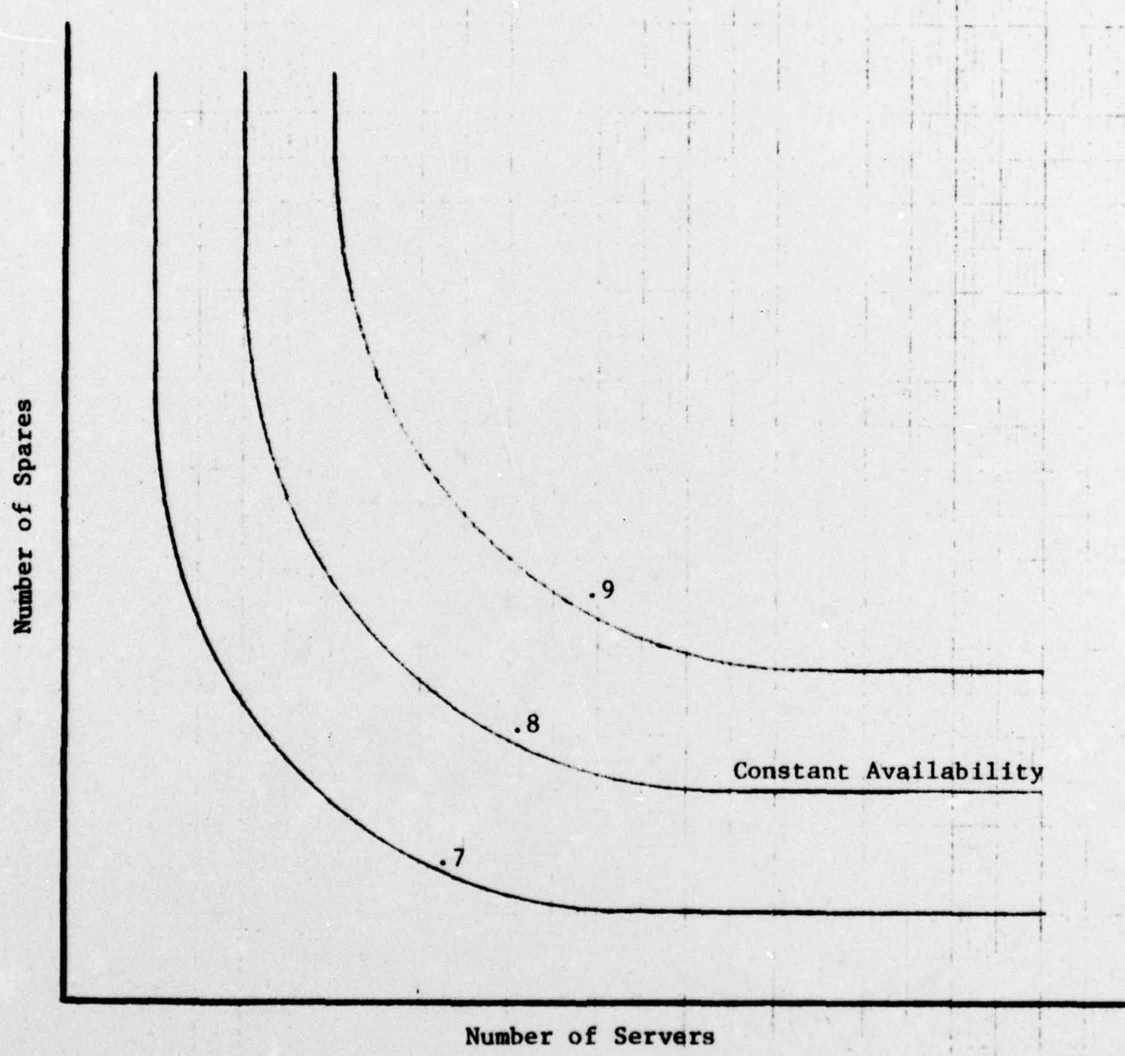

Figure 8.--Contours of availability vs. spares and servers.

"high" availability constraint, the approximate model was shown to be computationally more efficient and almost exact in its results. Furthermore, the approximate model was shown to be conservative, i.e., it always underestimated the true availability of the system.

A third approach to computing availability was investigated. This involved using the general time probabilities from the exact model (versus the exact failure point probabilities). The results were between the exact and approximate models in computation time and precision.

The main conclusion, then, is to use the approximate model if a "high" availability constraint is present, and the exact model otherwise. In fact, one may be forced to use the exact model for situations where $c_1\mu_1$ is not strictly less than $c_i\mu_i$ for $i=2,\ldots,K$ (which violates the basic assumption of an M/M/c queue). In those low availability cases where N is "large," the general time (versus failure point) probabilities for the exact model can be used to save computer time.

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APPENDIX A

Proof That $Q_E(n_1) \geq P_E(n_1) \geq Q_A(n_1)$ for $n_1 > c_1$

This appendix proves algebraically that the failure point probability in the exact model is greater than or equal to the general time probability in the exact model, which is greater than or equal to the failure point probability in the approximate model, for n_1 customers at Stage 1, when n_1 is greater than c_1, the number of servers at Stage 1.

Since $Q_A(n_1) = P_A(n_1)$ (i.e., the approximate model failure point and general time probabilities are equal), then we want to show

$$Q_E(n_1) \geq P_E(n_1) \geq P_A(n_1) \ , \quad \text{for } n_1 > c_1 \ , \tag{A1}$$

where, repeating relationships and notation here for convenience,

$Q_E(n_1)$ = failure point probability in the exact model that there are exactly n_1 customers at Stage 1

$$= A_3 c_1 \mu_1 P_E(n_1) \ ; \quad n_1 > c_1 \ . \tag{11}$$

$P_E(n_1)$ = general time probability in the exact model that there are exactly n_1 customers at Stage 1

$$= A_1 \sum_{S_2} \left\{ \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} \tag{10}$$

$P_A(n_1)$ = general time probability in the approximate model that there are exactly n_1 customers at Stage 1

$$= A_2 \sum_{S_2} \left\{ \prod_{i=2}^{K} \left(\frac{c_1 \mu_1}{\mu_i} \right)^{n_i} \frac{1}{b_i} \right\} \tag{17}$$

$$A_1^{-1} = \sum_{S_1} \left\{ \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i} \right)^{n_i} \frac{1}{b_i} \right\} \tag{7}$$

$$A_2^{-1} = \prod_{i=2}^{K} \left\{ \sum_{n_i=0}^{\infty} \left(\frac{c_1 \mu_1}{\mu_i} \right)^{n_i} \frac{1}{b_i} \right\} \tag{16}$$

$$A_3^{-1} = \sum_{n_1=0}^{c_1} n_1 \mu_1 \; P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1 \mu_1 \; P_E(n_1) \tag{12}$$

c_i = number of parallel servers at Stage i, $i=1,2,\ldots,K$

μ_i = service rate of each server at Stage i, $i=1,2,\ldots,K$

n_i = number of customers at Stage i, $i=1,2,\ldots,K$

K = total number of stages

N = total number of customers in the K stages = $\sum_{i=1}^{K} n_i$

$$b_i = \begin{cases} n_i! & ; \quad n_i \leq c_i \\ c_i! \; c_i^{\,n_i - c_i} & ; \quad n_i > c_i \end{cases} \tag{A2}$$

$$S_2 = \{n_i;\ i=2,3,\ldots,K : \sum_{i=2}^{K} n_i = N - n_1\} \tag{9}$$

$$S_1 = \{n_i;\ i=1,2,\ldots,K : \sum_{i=1}^{K} n_i = N\} . \tag{8}$$

With the notation established, consider first the inequality

$$Q_E(n_1) \overset{?}{\geq} P_E(n_1) \; ; \quad n_1 > c_1 \, .$$

Substituting (11) and (12),

$$\frac{c_1\mu_1 \; P_E(n_1)}{\sum_{n_1=0}^{c_1} n_1\mu_1 \; P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1\mu_1 \; P_E(n_1)} \overset{?}{\geq} P_E(n_1) \, .$$

Under the reasonable assumption that $P_E(n_1) \neq 0$ for $n_1=0,1,\ldots,N$, the above reduces to

$$c_1 \overset{?}{\geq} \sum_{n_1=0}^{c_1} n_1 P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1 P_E(n_1) \, .$$

Since $\sum_{n_1=0}^{N} P_E(n_1) = 1$, c_1 can be written as

$$c_1 = \sum_{n_1=0}^{c_1} c_1 P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1 P_E(n_1) \overset{?}{\geq} \sum_{n_1=0}^{c_1} n_1 P_E(n_1) + \sum_{n_1=c_1+1}^{N} c_1 P_E(n_1) \, ,$$

which reduces to

$$\sum_{n_1=0}^{c_1} c_1 P_E(n_1) \overset{?}{\geq} \sum_{n_1=0}^{c_1} n_1 P_E(n_1) \, ,$$

which is true since n_1 in the right-hand summation is always less than or equal to c_1 . Therefore,

$$\boxed{Q_E(n_1) \geq P_E(n_1) \; ; \quad n_1 > c_1}$$

Now it remains to prove the second part of Inequality (A1):

$$P_E(n_1) \geq P_A(n_1) \; ; \quad n_1 > c_1 \, .$$

Substituting (10) and (17),

$$A_1 \sum_{S_2} \left\{ \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} \overset{?}{\geq} A_2 \sum_{S_2} \left\{ \prod_{i=2}^{K} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} .$$

In order to get some cancellations, we change the index in the product term on the left-hand side and remove c_1 from the product term (noting that $\sum_{i=2}^{K} n_i = N-n_1$) on the right-hand side.

$$A_1 \sum_{S_2} \left\{ \frac{1}{b_1} \prod_{i=2}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} \overset{?}{\geq} A_2 \sum_{S_2} \left\{ c_1^{N-n_1} \prod_{i=2}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} .$$

On the left-hand side, using (A2),

$$b_1 = c_1!\, c_1^{n_1-c_1} \quad ; \quad n_1 > c_1 .$$

Under S_2 from (9), n_1 is a constant, and we can move terms outside the summations to get

$$\frac{A_1}{c_1!\, c_1^{n_1-c_1}} \left\{ \sum_{S_2} \prod_{i=2}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} \overset{?}{\geq} A_2\, c_1^{N-n_1} \left\{ \sum_{S_2} \prod_{i=2}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\} ,$$

or

$$\frac{A_1}{c_1!\, c_1^{N-c_1}} \overset{?}{\geq} A_2 .$$

Now substitute (7) and (16):

$$\frac{1}{c_1!\, c_1^{N-c_1} \sum_{S_2} \left\{ \prod_{i=1}^{K} \left(\frac{\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\}} \overset{?}{\geq} \frac{1}{\prod_{i=2}^{K} \left\{ \sum_{n_i=0}^{\infty} \left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i} \frac{1}{b_i} \right\}} .$$

Noting $c_1^N = c_1^{n_1} \cdot c_2^{n_2} \cdot \ldots \cdot c_1^{n_K}$, and by changing the index on the product term on the left-hand side, and by inverting both sides and changing the direction of the inequality,

$$\sum_{S_1}\left\{\frac{c_1!\ c_1^{n_1}}{c_1^{c_1}b_1}\prod_{i=2}^{K}\left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i}\frac{1}{b_i}\right\}\overset{?}{\leq}\prod_{i=2}^{K}\left\{\sum_{n_i=0}^{\infty}\left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i}\frac{1}{b_i}\right\}. \tag{A3}$$

Let

$$r=\frac{c_1!\ c_1^{n_1}}{c_1^{c_1}b_1}.$$

Since we are considering only $n_1 > c_1$, then by (A2), $b_1 = c_1!\ c_1^{n_1-c_1}$ and

$$r=\frac{c_1!\ c_1^{n_1}}{c_1^{c_1}\left(c_1!\ c_1^{n_1-c_1}\right)}=1\ .$$

Now letting

$$f_i=\left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i}\frac{1}{b_i},$$

we can write

$$\sum_{S_1}\left\{r\prod_{i=2}^{K}f_i\right\}=\sum_{S_1}\left\{\prod_{i=2}^{K}f_i\right\}\overset{?}{\leq}\prod_{i=2}^{K}\left\{\sum_{n_i=0}^{\infty}f_i\right\}.$$

It now only remains to show

$$\sum_{S_1}\left\{\prod_{i=2}^{K}f_i\right\}\overset{?}{\leq}\prod_{i=2}^{K}\left\{\sum_{n_i=0}^{\infty}f_i\right\}.$$

Under S_1 , the sum of n_i over all i is restricted to N . Since the f_i factor, $\left(\frac{c_1\mu_1}{\mu_i}\right)^{n_i}\frac{1}{b_i}$, is always positive, then

$$\sum_{S_1} \left\{ \prod_{i=2}^{K} f_i \right\} \leq \sum_{n_K=0}^{N} \sum_{n_{K-1}=0}^{N} \cdots \sum_{n_2=0}^{N} \left\{ \prod_{i=2}^{K} f_i \right\},$$

which simply bounds the summation over the restricted set S_1 to more manageable summations to N . Therefore, the final step is to show

$$\sum_{n_K=0}^{N} \cdots \sum_{n_3=0}^{N} \sum_{n_2=0}^{N} \left\{ \prod_{i=2}^{K} f_i \right\} \overset{?}{\leq} \prod_{i=2}^{K} \left\{ \sum_{n_i=0}^{\infty} f_i \right\}.$$

Substituting the value of f_i , the inequality to prove now is

$$\sum_{n_K=0}^{N} \cdots \sum_{n_2=0}^{N} \left(\frac{c_1\mu_1}{\mu_2}\right)^{n_2} \frac{1}{b_2} \left(\frac{c_1\mu_1}{\mu_3}\right)^{n_3} \frac{1}{b_3} \cdots \left(\frac{c_1\mu_1}{\mu_K}\right)^{n_K} \frac{1}{b_K}$$

$$\overset{?}{\leq} \left(\sum_{n_2=0}^{\infty} \left(\frac{c_1\mu_1}{\mu_2}\right)^{n_2} \frac{1}{b_2} \right) \cdot \left(\sum_{n_3=0}^{\infty} \left(\frac{c_1\mu_1}{\mu_3}\right)^{n_3} \frac{1}{b_3} \right) \cdots \left(\sum_{n_K=0}^{\infty} \left(\frac{c_1\mu_1}{\mu_K}\right)^{n_K} \frac{1}{b_K} \right).$$

It is clear that the product of infinite sums on the right-hand side will generate all the terms generated by the finite sums of the products on the left-hand side plus an infinite number more of miscellaneous cross products. Since all terms are positive, the inequality is satisfied. Therefore,

$$\boxed{P_E(n_1) \geq P_A(n_1) \; ; \quad n_1 > c_1} \; .$$

This completes the proof,

$$\boxed{Q_E(n_1) \geq P_E(n_1) \geq P_A(n_1) \; ; \quad n_1 > c_1} \; . \qquad \text{(A1)}$$

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